Iterative methods

Some definitions:

A is irreducible if there is no permutation P such that $P^T AP = \begin{bmatrix} A_{1,1} & A_{1,2} \\ 0 & A_{2,2} \end{bmatrix}$

A is non-negative (denoted as $A \geq 0$) if $a_{i,j} \geq 0$, for all $1 \leq i,j \leq n$

A is an M-matrix if A is nonsingular, $a_{i,j} \leq 0$ for $i \neq j$, and $A^{-1} \geq 0$

A is irreducibly diagonally dominant if A is irreducible,

diagonally dominant with $|a_{i,i}| > \sum_{j=1, j\neq i}^{n} a_{i,j}$ for some i.

$A=M-N$ is a regular splitting of A if M is nonsingular and $M^{-1} \geq 0$

Lemma 1: if $\|M\| < 1$ then $I - M$ is nonsingular

and $\|(I - M)^{-1}\| \leq \dfrac{1}{1 - \|M\|}$

Proof:Sketch of proof

1. $(I - M)^{-1} = I + M + M^2 + \ldots$

$$consider\ S_n = \sum_{k=0}^{n} M^k \quad for\ m > n$$

$$\|S_n - S_m\| \leq \sum_{k=n+1}^{m} \|M^k\| \leq \|M^{n+1}\| \cdot \frac{1 - \|M\|^{m-n}}{1 - \|M\|} \to 0,\ as\ m,\ n \to \infty$$

$$\Rightarrow S_n \to S$$

2. $I + M \cdot S_n = S_{n+1} \quad \Rightarrow \lim_{n\to\infty} I + M \cdot S_n = \lim_{n\to\infty} S_{n+1}$

$$\Rightarrow I + MS = S \quad \Rightarrow I = (I - M) \cdot S$$

$$\Rightarrow S = (I - M)^{-1} \quad \Rightarrow I - M\ is\ non\sin gular.$$

3. $\|(I - M)^{-1}\| \leq \sum_{k=1}^{\infty} \|M\|^k = \dfrac{1}{1 - \|M\|}$

Simple iterations:

$$x^{k+1} = Mx^k + c \quad \Rightarrow x^k \to x \quad when \ \left\| M \right\| < 1$$

$$proof : \ consider \ \begin{cases} x^{k+1} = Mx^k + c \\ x = Mx + c \end{cases} \Rightarrow \Delta x^{k+1} = M \Delta x^k$$

$$\Rightarrow \left\| \Delta x^{k+1} \right\| = \left\| (M)^k \Delta x_0 \right\| \le \left\| M \right\|^k \left\| \Delta x_0 \right\| \to 0 \quad as \ k \to \infty$$

Richardson iteration

$$x^{k+1} = x^k + \frac{1}{\alpha}(b - Ax^k) \Rightarrow x^{k+1} = \left(I - \frac{1}{\alpha} A \right) x^k + \frac{b}{\alpha}.$$

The iteration converges when $\left\| I - \frac{1}{\alpha} A \right\| < 1$

Suppose A is diagnalizable, $Q^T AQ = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$. Clearly when

$0 \le \alpha \le \dfrac{\max_i \lambda_i}{2}$, we have $\left\| I - \dfrac{1}{\alpha} A \right\| < 1$.

How many iterations needs to ensure error $< \varepsilon$?

$$\text{let } \alpha = \left\| \rho(A) \right\| = \max_i \lambda_i \quad \Rightarrow \left\| I - \frac{1}{\alpha} A \right\| = \left| 1 - \frac{\lambda_{\min}}{\lambda_{\max}} \right| = \left| 1 - \frac{1}{Cond(A)} \right|$$

$$\left(1 - \frac{1}{Cond(A)} \right)^k < \varepsilon \quad \Rightarrow k > \frac{\ln \varepsilon}{\ln\left(1 - \frac{1}{Cond(A)} \right)} \approx \mathrm{O}\left((\ln \varepsilon) Cond(A) \right)$$

Stationary Iterative Methods

1. $r^{old} = f - Au^{old}$
2. Solve $e = B^{-1} r^{old}$
3. update $u^{new} = u^{old} + e$

$\Leftrightarrow$

$$e^{new} = e^{old} - B^{-1}\left(f - Au^{old}\right)$$
$$= e^{old} - B^{-1}A\left(u - u^{old}\right)$$
$$= \left(I - B^{-1}A\right)e^{old}$$

B is called an iterator or preconditioner of A.

$E_B = I - B^{-1}A$ is called the error reduction operator of the iterator B

Perron-Frobenius Theorem

Theorem: Let $A \geq 0$ be an irreducible matrix. Then

1. A has a positive real eigenvalue equal to its spetral radius
2. There is an eigenvector $\mathbf{x} > 0$ corresponds to $\rho(A)$
3. $\rho(A)$ increases when any entry of A increases.
4. $\rho(A)$ is a simple eigenvalue of A.

Lemma 2 Let A and B be two matrices with $0 \le |B| \le A$. Then $\rho(B) \le \rho(A)$

Some Well Known Iterative Methods

Suppose $A = D - L - U$, where

D is the diagonal, L and U are lower and upper triangular parts, respectively.

Richardson: $B = \frac{1}{\omega}$, where $0 < \omega < \frac{2}{\rho(A)}$.

Jacobi: $B = D$

Damped Jacobi: $B = \frac{1}{\omega} D$, where $0 < \omega < \frac{2}{\rho(D^{-1}A)}$.

Gauss-Seidel: $B = (D - L)$

SOR: $B = \frac{1}{\omega}(D - \omega L)$, where $0 < \omega < 2$.

Matrix Splitting:

A matrix $A = M - N$ is a regular splitting
if M is nonsingular with $M^{-1} \geq 0$ and $N \geq 0$.

Theorem 1: Let $A = M - N$ be a regular splitting
then A is non-singular with $A^{-1} \geq 0$
iff $\rho(M^{-1}N) < 1$ where $\rho(M^{-1}N) = \dfrac{\rho(A^{-1}N)}{1+\rho(A^{-1}N)}$

proof: consider $M^{-1}A = I - M^{-1}N$ where $M^{-1} \geq 0, N \geq 0$

$$\Rightarrow M^{-1}N \geq 0$$

(1) Clearly, if $\rho(M^{-1}N) < 1$, then

$$A = \underbrace{M}_{\text{nonsingular}} \underbrace{(I - M^{-1}N)}_{\text{nonsingular}} \text{ is nonsingular}$$

(2) $A^{-1} = (I - M^{-1}N)^{-1} M^{-1}$

Since $\rho(M^{-1}N) < 1$, $I - M^{-1}N$ is nonsingular (by Lemma 1)

moreover $(I - M^{-1}N)^{-1} = I + (M^{-1}N) + (M^{-1}N)^2 + \ldots\ldots \geq 0$,

because $M^{-1}N \geq 0$. Clearly, $A^{-1} \geq 0$

$Stationary\ iterations,\ A = M - N\ be\ a\ regular\ splitting$

$$(*)\ x^{k+1} = x^k + M^{-1} r^k \quad where\ r^k = b - Ax^k$$

$$\Rightarrow e^{New} = (I - M^{-1}A) = (M^{-1}N)e^{old}$$

$Corollary:$

1. If A is nonsingular and $A^{-1} \geq 0$ then $(*)$ converges.

$$\bigcup$$

2. If a matrix A is an M-matrix, then $(*)$ converges.

$$\bigcup$$

3. If A is irreducibley diagonally dominant with

   $a_{ij} \leq 0,\ i \neq j,$ and $a_{ii} > 0,$ for all i. then A is an M-matrix.

Jacobi and Gauss-Seidel

Jacobi: $$x_i^{(m+1)} = -\sum_{\substack{j=1 \\ j \neq i}}^{n} \left(\frac{a_{i,j}}{a_{i,i}} \right) x_j^{(m)} + \frac{r_i}{a_{i,i}}$$

Gauss-Seidel: $$x_i^{(m+1)} = -\sum_{j=1}^{i-1} \left(\frac{a_{i,j}}{a_{i,i}} \right) x_j^{(m+1)} - \sum_{j=i+1}^{n} \left(\frac{a_{i,j}}{a_{i,i}} \right) x_j^{(m)} + \frac{r_i}{a_{i,i}}$$

HW1: Write down a formula for SOR

HW2:

Write a program to solve $$\begin{bmatrix} 1 & 0 & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 1 & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & 1 & 0 \\ -\frac{1}{4} & -\frac{1}{4} & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0.5 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$ by

Jacobi and Gauss-Seidel, starting with initial $\mathbf{x}^{(0)} = [0,0,0,0]$.

Let $E_J = \left(I - D^{-1}A\right)$ and $E_{GS} = \left(I - (D-L)^{-1}A\right)$. Since the solution of HW2 is $\mathbf{x}=[1,1,1,1]$ and $e^0 = x - x^{(0)} = [1,1,1,1]$. Clearly, we have $e_J^m = (E_J)^m e^0$ and $e_{GS}^m = (E_{GS})^m e^0$. One can easily check that

$$e_J^m = \frac{-1}{2^m}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \text{ and } e_{GS}^m = \frac{-1}{4^m}\begin{bmatrix}2\\2\\1\\1\end{bmatrix}. \text{ Thus, } \|e_J^m\| = \frac{1}{2^{m-1}} > \|e_{GS}^m\| = \frac{\sqrt{10}}{4^m}.$$

You might get a feeling that Gauss-Seidel method is faster than Jacobi method.

Stein-Rosenberg Theorem

Theorem: Let $B_J = L + U$ be the Jacobi matrix and $B_{GS} = (I-L)^{-1}U$ be the Gauss-Seidel matrix. Then one and only one of the following relations is vaild:

1) $\rho(B_J) = \rho(B_{GS}) = 0.$
2) $0 < \rho(B_{GS}) < \rho(B_J) < 1.$
3) $\rho(B_J) = \rho(B_{GS}) = 1.$
4) $1 < \rho(B_J) < \rho(B_{GS}).$

Convergence of Jacobi, Gauss-Seidel and SOR Iterative Methods

Lemma 1. If $A=(a_{i,j}) \geq 0$ is irreducible then either $\sum_{j=1}^{n} a_{i,j}=\rho(A)$ or

$$\min_{1\leq i\leq n}\left(\sum_{j=1}^{n} a_{i,j}\right) < \rho(A) < \max_{1\leq i\leq n}\left(\sum_{j=1}^{n} a_{i,j}\right) \quad \text{--------} \ (1).$$

Proof: Case(1): All row sums of A are equal ($=\sigma$): Let $\zeta=[1,1,\cdots,1]$. Clearly, $A\zeta=\sigma\zeta$ and $\sigma \leq \rho(A)$.

However, the Gerchgorin's Theorem implies $\rho(A) \leq \sigma$. Hence, $\rho(A) = \sigma$.

Case(2): Not all row sums of A are equal:

Construct $B=(b_{i,j}) \geq 0$ and $C=(c_{i,j}) \geq 0$, by decreasing and increasing some entries of A, respectively, such that

$$\sum_{j=1}^{n} b_{\ell,j} = \alpha = \min_{1\leq i\leq n}\left(\sum_{j=1}^{n} a_{i,j}\right) \text{ and } \sum_{j=1}^{n} c_{\ell,j} = \beta = \max_{1\leq i\leq n}\left(\sum_{j=1}^{n} a_{i,j}\right), \text{ for all } 1 \leq \ell \leq n.$$

By Perron-Frobenius theorem, we have $\rho(B) \leq \rho(A) \leq \rho(C)$.

Clearly, from the result of Case(1), the inequality (1) holds.

Theorem 1. Let $A=\left(a_{i,j}\right)$ be a strictly or irreducibly diagonally dominant matrix then the Jacobi and Gauss-Seidel iterative methods converge.

Proof: Recall that $E_J = I-D^{-1}A = D^{-1}(L+U) = \left(b_{i,j}\right)$, where $b_{i,j} = \begin{cases} 0 & i=j \\ \dfrac{-a_{i,j}}{a_{i,i}} & i \neq j \end{cases}$. From Lemma 2, it is clear that $\rho(B) \leq \rho(|B|)$. Since A is strictly diagonally dominant, clearly, we have $\sum_{j=1}^{n} \left|b_{i,j}\right| < 1$ for all $1 \leq i \leq n$. Therefore, Lemma 1 implies $\rho(|B|) < 1$. As a result, we have shown the Jacobi iterative method converge from $\rho(B) \leq \rho(|B|) < 1$.

Together with Theorem at p.35, this shows a strictly (or irreducibly) diagonally dominant matrix is an M-matrix.

Theorem 2. Let $A=D-E-E^*$ and D be Hermitian matrices, where D is positive definite, and $D-\omega E$ is non-singular for $0 \le \omega \le 2$.

Let $E_{SOR} = I-\omega(D-\omega E)^{-1}A$. Then $\rho(E_{SOR})<1$ if only if A is positive definite and $0<\omega<2$.

Proof: First, assume e_0 is a nonzero vector, the SOR iteration can be written as

$$(D-\omega E)e_{m+1} = (\omega E^* + (1-\omega)D)e_m,\ m \ge 0 \text{ ------- (2)}$$

Let $\delta_m = e_m - e_{m+1}$. Substracting $(D-\omega E)e_m$ and $(\omega E^* + (1-\omega)D)e_{m+1}$ from both side of (2),

we have $(D-\omega E)\delta_m = \omega A e_m$ ---- (3) and $\omega A e_{m+1} = [(1-\omega)D + \omega E^*]\delta_m$ ----- (4).

From $e_m^* \times (3) - e_{m+1}^* \times (4)$ and "simplifying the expression" (HW), one has

$$(2\text{-}\omega)\delta_m^* D\delta_m = \omega\{e_m^* A e_m - e_{m+1}^* A e_{m+1}\} \text{ ------ (5).}$$

Assume A is positive definite and $0<\omega<2$ and let e_0 be any eigenvector of E_{SOR}. We have $e_1=\lambda e_0$ and $\delta_0=(1\text{-}\lambda)e_0$ and (5) reduces to

$$\left(\frac{2\text{-}\omega}{\omega}\right)|1-\lambda|^2 e_0^* D e_0 = \left(1-|\lambda|^2\right)e_0^* A e_0 \text{ ------- (6).}$$

Now, $\lambda \ne 1$. Otherwise, $\delta_0 = 0 \Rightarrow Ae_0 = 0$ (by (3)) $\Rightarrow$ $e_0 = 0 \Rightarrow$ contradiction!

Since A and D are positive definite and $0<\omega<2$, (6) implies $1-|\lambda|^2>0$. Therefore, $\rho(E_{SOR})<1$.

Using similar arguments, one can show that the converse is also true.

Exercise: Together with Theorem at p.35, shows positive definite Hermitian matrices $[a_{ij}]$ with $a_{ij} \le 0,\ i \ne j,$ are M-matrices.